ON SOME PROPERTIES OF POLYNOMIALS.

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IT was shown by me in the ANALYST for Sept., 1879, that if a single series of equidistant terms, unlimited in length, such as

$$\ldots . u_{-2}, u_{-1}, u_0, u_1, u_2, u_3, \ldots .,$$

is adjusted by an unsymmetrical formula, for example

$$\begin{aligned} u'_0 = l_0u_0 &+ l_1 u_1 + l_2 u_2 + l_3 u_3 + l_4 u_4 \\ &+ l_{-1}u_{-1} + l_{-2}u_{-2} + l_{-3}u_{-3} + l_{-4}u_{-4}, \end{aligned} \tag{1}$$

thus affording the adjusted terms

$$\ldots . u'_{-2}, u'_{-1}, u'_0\ u'_1, u'_2, u'_3, \ldots .,$$

in such manner that

$$\begin{aligned} u'_1 &= l_0u_1 + l_1 u_2 + l_2 u_3 + l_3 u_4 + l_4 u_5 \\ &\quad + l_{-1}u_0 + l_{-2}u_{-1} + l_{-3}u_{-2} + l_{-4}u_{-3}, \\ u'_2 &= l_0u_2 + l_1 u_3 + l_2 u_4 + l_3 u_5 + l_4 u_6 \\ &\quad + l_{-1}u_1 + l_{-2}u_0 + l_{-3}u_{-1} + l_{-4}u_{-2}, \\ &\&c., \qquad \&c., \qquad \&c., \end{aligned}$$

and if this new series is adjusted again by another unsymmetrical formula, for example

$$u''_0 = L_0u'_0 + L_1u'_1 + L_2u'_2 + L_{-1}u'_{-1} + L_{-2}u'_{-2}, \tag{2}$$

the repeated adjustment thus obtained is equivalent to one adjustment of the original series by a resultant formula in which the adjusted term u'' is expressed in linear terms of u_0, u_1, u_2, &c., the coefficients of these last being sums of products of L and l. For instance, the term u_2 enters the resultant formula with the coefficient

$$L_{-2}l_4 + L_{-1}l_3 + L_0l_2 + L_1l_1 + L_2l_0. \tag{3}$$

Let us look a little more closely at the way in which this happens. In u'_{-2}, u'_{-1}, u'_0, u'_1, u'_2, &c., the term u_2 enters with the coefficients l_4, l_3, l_2, l_1, l_0, &c., respectively. In u''_0 the terms u'_{-2}, u'_{-1}, u'_0, u'_1, u'_2, enter with the coefficients L_{-2}, L_{-1}, L_0, L_1, L_2, respectively. Hence, u_2 enters into u''_0 through u'_{-2}, u'_{-1}, u'_0, u'_1, and u'_2 with the coefficients $L_{-2}l_4$, $L_{-1}l_3$, L_0l_2, L_1l_1 and L_2l_0 respectively, the sum of the sub-indices of the L and l in each of these coefficients being 2, the sub-index of u_2. The total influence of u_2 upon u''_0 is proportioned to the sum of all these coefficients as seen in (3).

Thus the resultant of two adjustments by any given component formulas similar to (1) and (2), no matter how many terms they may contain, will be a

formula in which any term u_r of the original series enters with a coefficient made up of the sum of all the products which can be formed by multiplying the L into the l in such manner that the sum of their sub-indices shall be r. And since it makes no difference whether we multiply an L into an l, or an l into an L, the resultant formula will be unchanged if we suppose the order of the operations to be reversed, so that the series is adjusted first by (2) and then by (1).

This mode of combining the coefficients L and l is precisely the same as that which occurs when we multiply together the two polynomials

$$l_{-4}z^{-4} + l_{-3}z^{-3} + l_{-2}z^{-2} + l_{-1}z^{-1} + l_0 z^0 + l_1 z + l_2 z^2 + l_3 z^3 + l_4 z^4,$$
$$L_{-2}z^{-2} + L_{-1}z^{-1} + L_0 z^0 + L_1 z + L_2 z^2,$$

where the indices of z are the same as the sub- indices of l and L, and the same also as those of u in (1) and (2). In the product, any power of z, for example z^2, will have for its coefficient the sum of all those products of L and l which have the sum of their sub-indices equal to 2, the index of the power of z under consideration. So too if we increase all the indices of z in the first factor by any number, for instance 4, and likewise increase all those of the second factor by any number, for instance 2, the product will be unchanged, except that all the indices of z in it will be increased by $4+2=6$. The coefficient of z^{2+6}, for instance, will be formed by those same combinations of L with l which formed the coefficient of z^2 in the previous case. Thus we establish the proposition, that any two component formulas of adjustment

$$\left.\begin{array}{rl} u'_0 = & l_0 u_0 + l_1 u_1 + l_2 u_2 + \ldots + l_m u_m \\ & + l_{-1}u_{-1} + l_{-2}u_{-2} + \ldots + l_{-m}u_{-m}, \\ u'_0 = & L_0 u_0 + L_1 u_1 + L_2 u_2 + \ldots + L_n u_n \\ & + L_{-1}u_{-1} + L_{-2}u_{-2} + \ldots + L_{-n}u_{-n}, \end{array}\right\} \qquad (4)$$

no matter which is used first, are equivalent to a single resultant formula, whose coefficients are the same and in the same order as those of the powers of z in the product of the polynomials

$$\left.\begin{array}{l} l_{-m} + l_{-m+1}z + l_{-m+2}z^2 + \ldots + l_0 z^m + l_1 z^{m+1} + \ldots + l_m z^{2m}, \\ L_{-n} + L_{-n+1}z + L_{-n+2}z^2 + \ldots + L_0 z^n + L_1 z^{n+1} + \ldots + L_n\ {}^{2n}. \end{array}\right\} \qquad (5)$$

The resultant cofficients of u_0, u_1, u_2, &c., are the same as those of z^{m+n}, z^{m+n+1}, z^{m+n+2}, &c., respectively. The whole number of terms in the resultant is $2(m+n)+1$.

Combinations analogous to these are found to arise in the repeated adjustment of a double series, or table of values of a function of two variables. The terms of the series are supposed to be arranged in rectangular form, as in (6), and each of them has two sub-indices, the first denoting its rank in the horizontal direction, reckoned from an initial term $u_{0,0}$, and the second

$u_{-1,2}$	$u_{0,2}$	$u_{1,2}$	$u_{2,2}$
$u_{-1,1}$	$u_{0,1}$	$u_{1,1}$	$u_{2,1}$
$u_{-1,0}$	$u_{0,0}$	$u_{1,0}$	$u_{2,0}$
$u_{-1,-1}$	$u_{0,-1}$	$u_{1,-1}$	$u_{2,-1}$

(6)

its rank in the vertical direction. The series may be extended without limit in both directions. For an investigation of some simple symmetrical formulas for the adjustment of such series, see the *Smithsonian Reports* of 1871 and 1873, pp. 321 and 332; an error being corrected in the ANALYST, May ,1877, p. 84. We will now consider the adjustment formulas in the most general way, as unsymmetrical, the terms u having coefficients l which may be any numbers, subject only to the condition that their algebraic sum is unity. Each l will have the same two sub-indices as the u to which it originally belongs.

For the sake of simplicity, suppose that a first adjustment is made by the nine term formula

$$u'_{0,0}=\left\{\begin{array}{l} l_{-1,1}\,u_{-1,1}+l_{0,1}\,u_{0,1}+l_{1,1}\,u_{1,1} \\ +l_{-1,0}\,u_{-1,0}+l_{0,0}\,u_{0,0}+l_{1,0}\,u_{1,0} \\ +l_{-1,-1}u_{-1,-1}+l_{0,-1}u_{0,-1}+l_{1,-1}u_{1,-1}\end{array}\right\}.$$

To make the notation more compact, we will designate any such rectangular formula by means of the terms in its four angles only, thus;

$$u'_{0,0}=\frac{l_{-1,1}\,u_{-1,1}}{l_{-1,-1}u_{-1,-1}}\left|\frac{l_{1,1}\,u_{1,1}}{l_{1,-1}u_{1,-1}}\right.. \tag{7}$$

It gives an unlimited number of adjusted terms,

$$u'_{1\ 0}=\frac{l_{-1,1}\,u_{0,1}}{l_{-1,-1}u_{0,-1}}\left|\frac{l_{1,1}\,u_{2,1}}{l_{1,-1}u_{2,-1}}\right.,\ u'_{1,1}=\frac{l_{-1,1}\,u_{0,2}}{l_{-1,-1}u_{0,0}}\left|\frac{l_{1,1}\,u_{2,2}}{l_{1,-1}u_{2,0}}\right.,\ \text{\&c., \&c.}$$

Let these be adjusted again by another formula, for instance

$$u''_{0,0}=\frac{L_{-2,1}\,u'_{-2,1}}{L_{-2,-1}u'_{-2,-1}}\left|\frac{L_{2,1}\,u'_{2,1}}{L_{2,-1}u'_{2,-1}}\right., \tag{8}$$

which includes 15 terms in three horizontal rows or five columns. A particular term in the original series (6), as $u_{2,1}$ for instance, can only enter into formula (8) through the adjusted values of the nine terms which have $u_{2,1}$ in their middle, and which we will denote by their corner terms only,

$$\begin{array}{c}u'_{1,2}\ \vdots\ u'_{3,2} \\ \cdots\cdots\cdots \\ u'_{1,0}\ \vdots\ u'_{3,0}\end{array}.$$

It would enter through these nine terms respectively with the nine coefficients

$$\begin{array}{l:l} L_{1,\,2}\,l_{1,\,-1} & L_{3,\,2}\,l_{-1,\,-1} \\ \hdashline L_{1,\,0}\,l_{1,\,1} & L_{3,\,0}\,l_{-1,\,1} \end{array}, \tag{9}$$

but for the fact that the first or horizontal sub-index of L in (8) nowhere exceeds 2, and the second or vertical one nowhere exceeds 1. Thus the five coefficients which contain $L_{1,\,2}$, $L_{2,\,2}$, $L_{3,\,2}$, $L_{3,\,1}$ and $L_{3,\,0}$ are all zero, and $u_{2,\,1}$ really enters (8) with only the four remaining coefficients from (9), its total influence upon $u''_{0,\,0}$ being proportioned to their sum

$$L_{1,\,1}\,l_{1,\,0}+L_{2,\,1}\,l_{0,\,0}+L_{1,\,0}\,l_{1,\,1}+L_{2,\,0}\,l_{0,\,1}.$$

In each of these products Ll the sum of the two horizontal sub-indices is 2, and the sum of the two vertical ones is 1, being the same as in $u_{2,\,1}$. Thus it appears that the resultant of any two component formulas such as (7) and (8), no matter how many terms they may contain, will be a formula in which any term $u_{r,\,s}$ of the original series enters with a coefficient made up of the sum of all the products which can be formed by multiplying the L into the l, in such manner that in each product the sum of the two horizontal sub-indices shall be r, and the sum of the two vertical ones shall be s. And as in the case of a single series, so here, the result will be unchanged if we reverse the order of operation, and adjust the double series first by (8) and then by (7).

Now let each u be replaced by the product of two variables, x and y, giving to the x an index or exponent equal to the horizontal sub-index of u, and to the y an index equal to the vertical sub-index of u. Then any two adjustment formulas, such as (7) and (8) for instance, will be respectively converted into the polynomials

$$\frac{l_{-1,\,1}\;x^{-1}y \;\big|\; l_{1,\,1}\;xy}{l_{-1,\,-1}x^{-1}y^{-1} \;\big|\; l_{1,\,-1}xy^{-1}},\quad \frac{L_{-2,\,1}\;x^{-2}y \;\big|\; L_{2,\,1}\;x^{2}y}{L_{-2,\,-1}x^{-2}y^{-1} \;\big|\; L_{2,\,-1}x^{2}y^{-1}}, \tag{10}$$

If these two are multiplied together, the coefficient of a term in the product, that of x^2y for example, will be just the same as that of the correrponding term $u_{2,\,1}$ in the resultant formula of adjustment, for it will be the sum of all those products of the L and l which can be formed so as to make the sum of the two horizontal or x indices equal to 2, and the sum of the two vertical or y indices equal to 1. If we increase all the indices of x and y in the first polynomial by any two numbers, for instance by 1 and 1 respectively, and increase those of the second polynomial by any two numbers, for instance 2 and 1 respectively, the product will be unchanged, except that all the indices of x will be increased by $1+2=3$, and all those of y by $1+1=2$. This proposition then is established, that any two component formulas of adjustment,

$$u'_{0,\,0}=\frac{l_{-m,\,n}\;u_{-m,\,n}\;\big|\;l_{m,\,n}\;u_{m,\,n}}{l_{-m,\,-n}u_{-m,\,-n}\big|\;l_{m,\,-n}u_{m,\,-n}},\quad u'_{0,\,0}=\frac{L_{-p,\,q}\;u_{-p,\,q}\;\big|\;L_{p,\,q}\;u_{p,\,q}}{L_{-p,\,-q}u_{-p,\,-q}\big|\;L_{p,\,-q}u_{p,\,-q}},\quad(11)$$

no matter which is used first, are equivalent to a single resultant formula whose coefficients are the same and in the same order as those of the products of the powers of x and y in the product of the two polynomials

$$\frac{l_{-m,\,n}\;x^0y^{2n}\;\big|\;l_{m,\,n}\;x^{2m}y^{2n}}{l_{-m,\,-n}x^0y^0\;\big|\;l_{m,\,-n}x^{2m}y^0},\qquad\frac{L_{-p,\,q}\;x^0y^{2q}\big|\;L_{p,\,q}\;x^{2p}\,y^{2q}}{L_{-p,\,-q}x^0y^0\;\big|\;L_{p,\,-q}x^{2p}\;y^0}.\qquad(12)$$

The resultant coefficients of $u_{0,\,0}$, $u_{1,\,0}$, $u_{1,\,1}$, &c., are respectively the same as those of

$$x^{m+p}\,y^{n+q},\quad x^{m+p+1}\,y^{n+q},\quad x^{m+p+1}\,y^{n+q+1},\quad \&c.$$

The whole number of terms in the component formulas being respectively $(2m+1)(2n+1)$ and $(2p+1)(2q+1)$, the whole number in the resultant is

$$[2(m+p)+1]\,[2(n+q)+1].\qquad(13)$$

It follows from the foregoing that if a double series is adjusted k times in succession by a single formula

$$u'_{0.\,0}=\frac{\lambda_{-m,\,n}\;u_{-m,\,n}\;\big|\;\lambda_{m,\,n}\;u_{m,\,n}}{\lambda_{-m,\,-n}u_{-m,\,-n}\big|\;\lambda_{m,\,-n}u_{m,\,-n}},\qquad(14)$$

and if the equivalent or resultant formula is denoted by

$$u^{(k)}_{0,\,0}=\frac{l_{-km,\,kn}\;\;u_{-km,\,kn}\;\;\big|\;l_{km,\,kn}\;\;u_{km,\,kn}}{l_{-km,\,-kn}u_{-km.\,-kn}\big|\;l_{km,\,-kn}u_{km,\,-kn}},\qquad(15)$$

the coefficients l will be the same as the coefficients in the expansion of the polynomial

$$\left\{\frac{\lambda_{-m,\,n}\;\;x^0y^{2n}\big|\;\lambda_{m,\,n}\;\;x^{2m}y^{2n}}{\lambda_{-m.\,-n}x^0y^0\;\big|\;\lambda_{m,\,-n}x^{2m}y^0}\right\}^k.\qquad(16)$$

The number of terms in the expansion is $(2km+1)(2kn+1)$.

It was shown in the Analyst for Jan., 1880, p. 2, that if the coefficients in a polynomial of one variable,

$$a_0+a_1z+a_2z^2+\;.\;.\;.\;+a_nz^n,$$

are regarded as a system of equidistant parallel forces acting at right angles to the axis of X taken as a mathematical lever, and if h_1 denotes the distance from the centre of parallel forces to the point of application of a_0, or in other words, the lever arm of the system about the place of a_0 as a fulcrum, then in the k power of this polynomial the lever arm of the system of coefficients about the place of the first one as a fulcrum will be kh_1. This can be proved in a manner rather more simple and general, and without introducing the condition

$$a_0+a_1+a_2+\;.\;.\;.\;+a_n=1,$$

which holds when a_0, a_1, &c., represent coefficients in a formula of adjustment. Let any two polynomials in z be

$$\left.\begin{array}{l} a_0+a_1z+a_2z^2+\ldots+a_mz^m, \\ c_0+c_1z+c_2z^2+\ldots+c_nz^n, \end{array}\right\} \tag{17}$$

and let the sums of their coefficients be

$$\left.\begin{array}{l} S_1=a_0+a_1+a_2+\ldots+a_m, \\ S_2=c_0+c_1+c_2+\ldots+c_n. \end{array}\right\} \tag{18}$$

Regarding these as systems of equidistant parallel forces, let the lever arms of the two systems about the points of application of a_0 and c_0 be h_1 and h_2, respectively, and in each system let Δx be the interval between the consecutive points of application of the forces. The moments of the two systems about the places of a_0 and c_0 will be

$$\left.\begin{array}{l} S_1h_1=(a_1+2a_2+3a_3+\ldots+ma_m)\Delta x, \\ S_2h_2=(c_1+2c_2+3c_3+\ldots+nc_n)\Delta x. \end{array}\right\} \tag{19}$$

When the two polynominls 17 are multiplied together, the moment of the coefficients of that part of the product which is due to c_0, about the place of a_0c_0, will be $c_0S_1h_1$. The moment of those of the part which is due to c_1z will be

$$c_1S_1(h_1+\Delta x),$$

for the part due to c_2z^2 it will be

$$c_2S_1(h_1+2\Delta x),$$

and so on. Denoting by H the lever arm of the whole system of coefficients in the product, we shall have for the moment of this system about the place of the first term as a fulcrum,

$$\begin{aligned} S_1S_2H &= c_0S_1h_1+c_1S_1(h_1+\Delta x)+c_2S_1(h_1+2\Delta x)+\ldots+c_nS_1(h_1+n\Delta x) \\ &= S_1h_1(c_0+c_1+\ldots+c_n)+S_1(c_1+2c_2+\ldots+nc_n)\Delta x. \end{aligned}$$

By (18) and (19) this reduces to

$$S_1S_2H=S_1S_2h_1+S_1S_2h_2,$$

$$\therefore\ H=h_1+h_2, \tag{20}$$

so that the lever arm of the coefficients in the product is equal to the sum of the lever arms of those in the two factors. From this it follows that if any number of polynomials are multiplied together, the lever arm of the coefficients in the final product, about its first term as a fulcrum, is equal to the sum of the lever arms in all the factors, about their first terms as fulcrums. Hence, if the first polynomial in (17) is to be raised to the k power, the lever arm of the coefficients in the expansion of

$$(a_0+a_1z+a_2z^2+\ldots+a_mz^m)^k$$

will be equal to kh_1. Now the whole length of the polynomial is $m\Delta x$, and the whole length of its expansion to the k power is $km\Delta x$. Thus it is proved that as k increases from one integer to another, the lever arm of the product increases in the same ratio as the whole length of the product does,

so that the extremity of the lever arm, or the centre of parallel forces, divides the length of the expanded polynomial in a constant ratio. If the coefficients are all positive, they may be regarded as masses in a system of material points, and the centre of forces becomes the centre of gravity.

Analogous properties are found to belong to polynomials of two variables, whose terms are supposed to be arranged thus,

$$\begin{array}{llll} + \ldots & + \ldots & + \ldots & + \ldots \\ +a_{0,2}y^2 & +a_{1,2}xy^2 & +a_{2,2}x^2y^2 + & \ldots \\ +a_{0,1}y & +a_{1,1}xy & +a_{2,1}x^2y + & \ldots \\ +a_{0,0} & +a_{1,0}x & +a_{2,0}x^2 & + \ldots \end{array} \tag{21}$$

so that the indices of x and y in any term correspond to the rank of that term reckoned from the left hand column and the lower row respectively. Let any two such polynomials be denoted for brevity by

$$\frac{a_{0,n}\,x^0y^n \mid a_{m,n}\,x^my^n}{a_{0,0}\,x^0y^0 \mid a_{m,0}\,x^my^0}, \qquad \frac{c_{0,q}\,x^0y^q \mid c_{p,q}x^p\,y^q}{c_{0,0}\,x^0y^0 \mid c_{p,0}x^p\,y^0}, \tag{22}$$

so that the first one has $m+1$ and $n+1$ terms in its rows and columns, respectively, while the second one has $p+1$ and $q+1$. Let the sums of their coefficients be

$$S_1 = \frac{a_{0,n} \mid a_{m,n}}{a_{0,0} \mid a_{m,0}}, \qquad S_2 = \frac{c_{0,q} \mid c_{p,q}}{c_{0,0} \mid c_{p,0}}. \tag{23}$$

Regarding these coefficients as systems of equidistant parallel forces acting upon the plane of XY at right angles to it, let the lever arms of the two systems about the lower row or axis of X be h_1 and h_2, and let those about the left hand column or axis of Y be k_1 and k_2. Also in each system let Δx and Δy be the intervals between consecutive points of application of the forces, in the directions of X and Y. In the second system, let the sum of the coefficients in the first or lower row be s''_0, and let the sums of those in the other successive rows be s''_1, s''_2, s''_3, &c., wherefore

$$S_2 = s''_0 + s''_1 + \ldots + s''_q. \tag{24}$$

The moment of the second system about the axis of X is

$$S_2 h_2 = (s''_1 + 2s''_2 + \ldots + qs''_q)\Delta y. \tag{25}$$

When the two polynomials (22) are multiplied together, the moment of the coefficients of that part of the product due to the first row in the second polynomial, about the axis of X or place of the first row in the product, will be $s''_0 S_1 h_1$. The moment for the part due to the second row will be

$$s''_1 S_1 (h_1 + \Delta y).$$

In the part due to the third row the moment will be

$$s''_2 S_1 (h_1 + 2\Delta y),$$

and so on. Denoting by H and K the lever arms of the whole system of

coefficients in the product about the axes of X and Y, we have for the total moment of this system about the line of its lower row as a fulcrum,

$$\begin{aligned} S_1 S_2 H &= s_0'' S_1 h_1 + s_1'' S_1(h_1+\Delta y) + s_2'' S_1(h_1+2\Delta y) + \ldots + s_q'' S_1(h_1+q\Delta y) \\ &= S_1 h_1(s_0''+s_1''+\ \ldots\ +s_q'') + S_1(s_1''+2s_2''+\ \ldots\ +qs_q'')\Delta y. \end{aligned}$$

By (24) and (25) this reduces to

$$\begin{aligned} S_1 S_2 H &= S_1 S_2 h_1 + S_1 S_2 h_2, \\ \therefore\ H &= h_1 + h_2, \end{aligned} \tag{26}$$

showing that the lever arm of the coefficients in the product, about the line of its lower row, is equal to the sum of the lever arms of those in the two factors, about the lines of their lower rows. From the similarity of situation of the axes of X and Y with respect to the rectangular system of coefficients, it follows that we shall have in like manner

$$K = k_1 + k_2. \tag{27}$$

Hence, if any number of such polynomials in x and y are multiplied together, the lever arm of the system of coefficients in the final product, about the line of its lower row, is equal to the sum of the lever arms of the systems in all the factors, about the lines of their lower rows; and likewise the lever arm for the product, about the line of its left hand column, is equal to the sum of the lever arms for all the factors, about the lines of their left hand columns. Similar properties would evidently exist if the rotations were supposed to be about the line of the upper row or of the right hand column, for the coefficients in the factors being arbitrary, their positions may be reversed, and those of the product will be reversed also. Hence too, if a single polynomial, the first one in (22) for instance, is to be raised to the k power, the lever arms H and K of the coefficients in the expansion of

$$\left\{ \frac{a_{0,\,n}\,x^0 y^n \mid a_{m,\,n}\,x^m y^n}{a_{0,\,0}\,x^0 y^0 \mid a_{m,\,0}\,x^m y^0} \right\}^k \tag{28}$$

will be equal to kh_1 and kk_1 respectively. But the length and breadth of the given polynomial are $m\Delta x$ and $n\Delta y$, and the length and breadth of its expansion to the k power will be $km\Delta x$ and $kn\Delta y$. Thus it is proved that as k increases from one integer to another, the lever arms of the product increase in the same ratio as the dimensions of the product do. The centre of parallel forces in each case will be that point in the plane of XY whose distances from the axes of X and Y are h_1 and k_1 for the given polynomial, and kh_1 and kk_1 for its expansion to the k power, so that as k is increased, the position of this centre of forces, relatively to the four sides of the rectangle formed by the polynomial, will remain unchanged. If the coefficients are all positive, we may regard them as masses, and the centre of forces is their centre of gravity.

(To be continued.)